

On the temporal decay for the Hall-magnetohydrodynamic equations

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Abstract

We establish temporal decay estimates for weak solutions to the Hall-magnetohydrodynamic equations. With these estimates in hand we obtain algebraic time decay for higher order Sobolev norms of small initial data solutions.

Key words: Hall-MHD equations, temporal decay estimates, Fourier-Splitting method

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1 Introduction

We consider the incompressible MHD-Hall equations in \mathbb{R}^3 .

$$\partial_t u + u \cdot \nabla u + \nabla p = (\nabla \times B) \times B + \nu \Delta u, \quad (1.1)$$

$$\nabla \cdot u = 0, \quad \nabla \cdot B = 0, \quad (1.2)$$

$$\partial_t B - \nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) = \mu \Delta B, \quad (1.3)$$

$$u(x, 0) = u_0(x) \quad ; \quad B(x, 0) = B_0(x). \quad (1.4)$$

Here $u = (u_1, u_2, u_3) = u(x, t)$ is the velocity of the charged fluid, $B = (B_1, B_2, B_3)$ the magnetic field induced by the motion of the charged fluid, $p = p(x, t)$ the pressure of the fluid. The positive constants ν and μ are the viscosity and the resistivity coefficients. Without loss of generality we let $\mu = \nu = 1$. Compared with the usual viscous incompressible MHD system, the system (1.1)-(1.4) contains the extra term $\nabla \times ((\nabla \times B) \times B) = \nabla \times ((\nabla \times B) \times B)$, which is the so called Hall term. This term is important when the magnetic shear is large, where the magnetic reconnection happens. On the other hand, in the case of laminar flows where the shear is weak, one ignores the Hall term, and the system reduces to the usual MHD. We refer [4, 12] for the physical background of the magnetic reconnection and the Hall-MHD.

Compared to the case of the usual MHD the history of the fully rigorous mathematical study of the Cauchy problem for the Hall-MHD system is very short. The global existence of weak solutions in the periodic domain is done in [1] by a Galerkin approximation. The global existence in the whole domain in \mathbb{R}^3 as well as the local well-posedness of smooth solution is proved in [2], where the global existence of smooth solution for small initial data is also established.

In this paper we study the Cauchy problem of the Hall-MHD system and establish temporal decay estimates for the solutions. Our results, provide a mathematically rigorous basis to explain the decay of energy in the Hall-MHD, which had been obtained by numerical simulations (see e.g. [3, 6]).

Algebraic rates for the asymptotic behavior of solutions to the Navier-Stokes equations were obtained first by the second author of this paper in [9], using the method of Fourier Splitting. This technique was introduced first to study the decay of solutions to parabolic conservation laws [8]. The Fourier Splitting method was then refined in in [10, 11]. (see also [5]). Here we apply the arguments of [9] and [10] to obtain algebraic time decay rates

for the solutions of the Hall-MHD system. The existence of Hall term in the equations generates extra terms to control, which needed to be handled in our proofs by introducing new estimates. We now list the main theorems of the paper. The first is a preliminary decay estimate for the weak solutions.

Theorem 1.1 *Let $(u_0, B_0) \in (L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))^2$ with $\operatorname{div} B_0 = \operatorname{div} u_0 = 0$. Then, there exists a weak solution to the system (1.1)-(1.4), which satisfies*

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \leq C(t+1)^{-\frac{3}{2}}. \quad (1.5)$$

The following is a decay estimate for the higher order Sobolev norms, whose global in time existence is guaranteed for sufficiently small initial data([2]).

Theorem 1.2 *Let $(u_0, B_0) \in (L^1(\mathbb{R}^3))^2$ satisfies the conditions of Theorem 2.1 below. Assume that*

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \leq C_0(t+1)^{-2\mu} \quad (1.6)$$

for $t \geq 0$ and $\mu \geq 0$. Then, for $m \in \mathbb{N}$ there exists $C_m = C_m(\mu, C_0)$ and $T_ > 0$ such that*

$$\|D^m u(t)\|_{L^2}^2 + \|D^m B(t)\|_{L^2}^2 \leq C_m(t+1)^{-m-2\mu} \quad (1.7)$$

for all $t \geq T_$.*

Remark 1.1 Since (1.6) is valid for $\mu = 3/4$ by Theorem 1.1, we obtain the decay estimate,

$$\|D^m u(t)\|_{L^2}^2 + \|D^m B(t)\|_{L^2}^2 \leq C_m(t+1)^{-m-\frac{3}{2}} \quad (1.8)$$

for $t \geq T_$.*

2 Proof of the main theorems

We recall that to use the Fourier Splitting technique we need the following two main estimates: Let $V(\cdot, t) \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$,

$$\frac{d}{dt} \|V(t)\|_{L^2} \leq -C \|\nabla V(t)\|_{L^2}^2. \quad (2.9)$$

$$|\widehat{V}(\xi, t)| \leq C \quad \text{for } |\xi| \ll 1, \quad (2.10)$$

where \widehat{V} denotes the Fourier transform of V defined by

$$\widehat{V}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} V(x) e^{-ix \cdot \xi} dx, \quad i = \sqrt{-1}.$$

These conditions will insure that $\|V(t)\|_{L^2}$ decays at the same rate as the solutions of the heat equations, with the same data $V(x, 0)$, [7]. We will use this method with appropriate modifications for our equations. The next Lemma is in the spirit of condition (2.10) above.

Lemma 2.1 *Let (u, B) be a smooth solution to the system (1.1)-(1.4) with (u_0, B_0) satisfying the initial condition as in Theorem 1.1. Then, we have*

$$|\hat{u}(\xi, t)| + |\hat{B}(\xi, t)| \leq C \left(1 + \frac{1}{|\xi|} \right), \quad (2.1)$$

where $C = C(\|u_0\|_{L^1 \cap L^2} + \|B_0\|_{L^1 \cap L^2})$.

Proof Using the elementary vector calculus, one can rewrite (1.1) and (1.3) as

$$u_t - \Delta u = -\mathbb{P} \nabla \cdot (u \otimes u - B \otimes B), \quad (2.2)$$

and

$$B_t - \Delta B = -\nabla \cdot (u \otimes B - B \otimes u) - \nabla \times \{ \nabla \cdot (B \otimes B) \} \quad (2.3)$$

respectively, where \mathbb{P} is the Leray projection operator defined by $\mathbb{P}f = f - \nabla \Delta^{-1} \nabla \cdot f$. Hence, we have the following representation of solutions in terms of the Fourier transform,

$$\begin{aligned} \hat{u}(\xi, t) &= e^{-|\xi|^2 t} \hat{u}_0(\xi) \\ &\quad - \int_0^t e^{-|\xi|^2(t-s)} \left(1 - \frac{\xi \otimes \xi}{|\xi|^2} \right) \left\{ \xi \cdot \widehat{(u \otimes u)}(\xi, s) - \xi \cdot \widehat{(B \otimes B)}(\xi, s) \right\} ds, \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \hat{B}(\xi, t) &= e^{-|\xi|^2 t} \hat{B}_0(\xi) \\ &\quad - \int_0^t e^{-|\xi|^2(t-s)} \left\{ \xi \cdot \widehat{(u \otimes B)}(\xi, s) - \xi \cdot \widehat{(B \otimes u)}(\xi, s) - \xi \times \{ \xi \cdot \widehat{(B \otimes B)}(\xi, s) \} \right\} ds. \end{aligned} \quad (2.5)$$

From these representations we obtain

$$\begin{aligned}
|\hat{u}(\xi, t)| &\leq |\hat{u}_0(\xi)| + \int_0^t |\xi| e^{-|\xi|^2(t-s)} \{ |(\widehat{u \otimes u})(\xi, s)| + |(\widehat{B \otimes B})(\xi, s)| \} ds \\
&\leq \|u_0\|_{L^1} + \int_0^t |\xi| e^{-|\xi|^2(t-s)} (\|u(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2) ds \\
&\leq \|u_0\|_{L^1} + (\|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2) \frac{1}{|\xi|} (1 - e^{-|\xi|^2 t}) \\
&\leq C \left(1 + \frac{1}{|\xi|} \right), \tag{2.6}
\end{aligned}$$

and

$$\begin{aligned}
|\hat{B}(\xi, t)| &\leq |\hat{B}_0(\xi)| + \int_0^t e^{-|\xi|^2(t-s)} |\xi| \{ |(\widehat{u \otimes B})(\xi, s)| + |(\widehat{B \otimes u})(\xi, s)| \} ds \\
&\quad + \int_0^t e^{-|\xi|^2(t-s)} |\xi|^2 |(\widehat{B \otimes B})(\xi, s)| ds \\
&\leq \|B_0\|_{L^1} + \int_0^t e^{-|\xi|^2(t-s)} (2|\xi| \|u(s)B(s)\|_{L^1} + |\xi|^2 \|B(s)\|_{L^2}^2) ds \\
&\leq \|B_0\|_{L^1} + \left\{ \frac{2}{|\xi|} \|u_0\|_{L^2} \|B_0\|_{L^2} + \|B_0\|_{L^2}^2 \right\} (1 - e^{-|\xi|^2 t}) \\
&\leq C \left(1 + \frac{1}{|\xi|} \right). \tag{2.7}
\end{aligned}$$

□

Proof of Theorem 1.1 Multiplying (1.1) by u , and (1.3) by B , and integrating over \mathbb{R}^3 , and integrating by part we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} (|u|^2 + |B|^2) dx = -2 \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla B|^2) dx. \tag{2.8}$$

Using the estimates (2.8) and (2.1), and applying the standard Fourier-Splitting method, developed in [9] and [10], one can conclude the estimate (1.5). More specifically by (2.8) and (2.1), one obtains a preliminary decay estimate that allows, as was done for the Navier-Stokes equations, to obtain the better estimate on the Fourier transfer of (u, B) near the origin in frequency space, as required in condition (2.10). □

In order to prove Theorem 1.2 we first recall the following small data global regularity result proved in [2](see Theorem 2.3).

Theorem 2.1 *Let $m \in \mathbb{N}, m \geq 3$ and $(u_0, B_0) \in [H^m(\mathbb{R}^3)]^2$ with $\operatorname{div} B_0 = \operatorname{div} u_0 = 0$. There exists a constant K_1 such that if $\|u_0\|_{H^m} + \|B_0\|_{H^m} \leq K_1$, then there exists a unique solution $(u, B) \in L^\infty(\mathbb{R}_+; H^m(\mathbb{R}^3))$ satisfying*

$$\frac{d}{dt}(\|u\|_{H^m}^2 + \|B\|_{H^m}^2) \leq -(\|\nabla u\|_{H^m}^2 + \|\nabla B\|_{H^m}^2). \quad (2.9)$$

Remark 2.1 Although the global energy inequality (2.9) is not written down in [2], it is immediate by choosing the constant $K_1 = \frac{1}{2}K$, where K is the constant in Theorem 2.3([2]) bounding the initial data to obtain the global smooth solution.

We observe the following fact.

Lemma 2.2 *Let $(u_0, B_0) \in (L^1(\mathbb{R}^3))^2$ be as in Theorem 2.1. Then, for all $|\xi| \leq 1$ and for all $j \in \mathbb{N}$ we have*

$$|\widehat{D^j u}(\xi, t)| + |\widehat{D^j B}(\xi, t)| \leq C, \quad (2.10)$$

where $C = C(\|u_0\|_{L^1 \cap L^2} + \|B_0\|_{L^1 \cap L^2})$.

Proof Let $|\xi| \leq 1$. Using the result of Lemma 2.1, we have

$$\begin{aligned} |\widehat{D^j u}(\xi, t)| + |\widehat{D^j B}(\xi, t)| &\leq |\xi|^j(|\hat{u}(\xi, t)| + |\hat{B}(\xi, t)|) \\ &\leq |\xi|(|\hat{u}(\xi, t)| + |\hat{B}(\xi, t)|) \leq C|\xi| \left(1 + \frac{1}{|\xi|}\right) \leq C. \end{aligned}$$

□

The following is an auxiliary decay estimate for higher order Sobolev norms.

Theorem 2.2 *Let $(u_0, B_0) \in (L^1(\mathbb{R}^3))^2$ be as in Theorem 2.1. Then, there exists a constant C such that*

$$\|u(t)\|_{H^m}^2 + \|B(t)\|_{H^m}^2 \leq C(t+1)^{-\frac{3}{2}}. \quad (2.11)$$

Proof We apply the Fourier-Splitting method. Let $(\alpha_1, \alpha_2, \alpha_3) \in [\mathbb{N} \cup \{0\}]^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, be the multi-index. The Fourier transform of (2.9) is written as

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}|^2 \right) d\xi \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} \sum_{|\alpha| \leq m} (|\xi_1|^{\alpha_1} |\xi_2|^{\alpha_2} |\xi_3|^{\alpha_3})^2 (|\hat{u}|^2 + |\hat{B}|^2) d\xi \\ & \quad - \int_{\mathbb{R}^3} \sum_{|\alpha| \leq m} |\xi|^2 (|\xi_1|^{\alpha_1} |\xi_2|^{\alpha_2} |\xi_3|^{\alpha_3})^2 (|\hat{u}|^2 + |\hat{B}|^2) d\xi. \end{aligned} \quad (2.12)$$

Let

$$S := \left\{ \xi \in \mathbb{R}^3 \mid |\xi| \leq \left(\frac{k}{t+1} \right)^{\frac{1}{2}} \right\}.$$

From (2.12) it follows that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}|^2 \right) d\xi \\ & \leq - \int_S |\xi|^2 \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}|^2 \right) d\xi \\ & \quad - \int_{S^c} |\xi|^2 \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}|^2 t \right) d\xi \\ & \leq - \frac{k}{t+1} \int_{\mathbb{R}^3} \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}|^2 \right) d\xi \\ & \quad + \frac{k}{t+1} \int_S \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}|^2 \right) d\xi. \end{aligned} \quad (2.13)$$

Since $|\xi| \leq k(t+1)^{-\frac{1}{2}}$, and $|\widehat{D^\alpha u}(\xi, t)| + |\widehat{D^\alpha B}(\xi, t)| \leq C$ for $\xi \in S$ and $t \geq T_0$ for some $T_0 > 0$ by Lemma 2.2, we have

$$\begin{aligned} & \frac{d}{dt} \left[(1+t)^k \int_{\mathbb{R}^3} \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}|^2 \right) d\xi \right] \\ & \leq k(t+1)^{k-1} \int_S \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}|^2 \right) d\xi \\ & \leq C(t+1)^{k-\frac{5}{2}} \quad \forall t \geq T_0. \end{aligned} \tag{2.14}$$

Integrating over $[T_0, t]$, and dividing by $(t+1)^k$, we find

$$\begin{aligned} & \int_{\mathbb{R}^3} \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}(\xi, t)|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}(\xi, t)|^2 \right) d\xi \\ & \leq (t+1)^{-k} \int_{\mathbb{R}^3} \left(\sum_{|\alpha| \leq m} |\widehat{D^\alpha u}(\xi, T_0)|^2 + \sum_{|\alpha| \leq m} |\widehat{D^\alpha B}(\xi, T_0)|^2 \right) d\xi \\ & \quad + C(t+1)^{-\frac{3}{2}}. \end{aligned} \tag{2.15}$$

The estimate (2.11) follows if we choose $k = 3/2$. \square

In order to establish Theorem 1.4 we first show the following auxiliary lemma, which is similar in form to Lemma 3.2.

Lemma 2.3 *Let (u, B) is a smooth solution of (1.1)-(1.4) and $m \in \mathbb{N}$. Then, we have the following inequality.*

$$\begin{aligned} & \frac{d}{dt} (\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) + \|D^{m+1} u\|_{L^2}^2 + \|D^{m+1} B\|_{L^2}^2 \\ & \leq C_m (\|u\|_{L^\infty}^2 \|D^m u\|_{L^2}^2 + \|B\|_{L^\infty}^2 \|D^m B\|_{L^2}^2 + \|B\|_{L^\infty}^2 \|D^m u\|_{L^2}^2 \\ & \quad + \|u\|_{L^\infty}^2 \|D^m B\|_{L^2}^2 + \|\nabla B\|_{L^\infty}^2 \|D^m B\|_{L^2}^2) + R_m, \end{aligned} \tag{2.16}$$

where

$$R_m = \begin{cases} 0, & \text{if } m = 1, 2 \\ C_m \sum_{1 \leq j \leq m/2} (\|D^j u\|_{L^\infty}^2 \|D^{m-j} u\|_{L^2}^2 + \|D^j B\|_{L^\infty}^2 \|D^{m-j} B\|_{L^2}^2) \\ + C_m \sum_{1 \leq j \leq m/2} (\|D^j B\|_{L^\infty}^2 \|D^{m-j} u\|_{L^2}^2 + \|D^j u\|_{L^\infty}^2 \|D^{m-j} B\|_{L^2}^2) \\ + C_m \sum_{2 \leq i \leq \frac{m+1}{2}} \|D^i B\|_{L^\infty}^2 \|D^{m+1-i} B\|_{L^2}^2, & \text{if } m \geq 3. \end{cases} \quad (2.17)$$

Remark 2.2 Although the Hall term generates the extra factor of norm for the derivative of B such as $\|\nabla B\|_{L^\infty}$, which is not present in [11], this can be handled without difficulty as shown in the proof of Theorem 1.2 below.

Proof of Lemma 2.3 Let $m \geq 3$. Multiplying (1.1) and (1.3) by u and B respectively, and integrating each one over \mathbb{R}^3 , and integrating by part, we obtain the following inequalities.

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} |D^m u|^2 dx &\leq C \sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^m(u_k u_j)|^2 dx + C \sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^m(B_k B_j)|^2 dx \\ &\quad - \int_{\mathbb{R}^3} |D^{m+1} u|^2 dx, \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} |D^m B|^2 dx &\leq C \sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^m(B_k u_j)|^2 dx + C \sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^m(u_k B_j)|^2 dx \\ &\quad + C \int_{\mathbb{R}^3} |D^m(B \cdot \nabla) B|^2 dx - \int_{\mathbb{R}^3} |D^{m+1} B|^2 dx, \end{aligned}$$

where we used the fact $\|\nabla f\|_{L^2} = \|\nabla \times f\|_{L^2}$ if $\nabla \cdot f = 0$. We have the following auxiliary estimates. For $m \geq 2$

$$\begin{aligned} \sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^m(u_k u_j)|^2 dx &\leq C \sum_{j,k=1}^3 \sum_{1 \leq i \leq m/2} \int_{\mathbb{R}^3} |D^{m-i} u_k D^i u_j|^2 dx + C \int_{\mathbb{R}^3} |D^m u|^2 |u|^2 dx \\ &\leq C \sum_{1 \leq i \leq m/2} \|D^i u\|_{L^\infty}^2 \|D^{m-i} u\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \|D^m u\|_{L^2}^2. \end{aligned} \quad (2.19)$$

Similarly, we have

$$\sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^m(B_k B_j)|^2 dx \leq C \sum_{1 \leq i \leq m/2} \|D^i B\|_{L^\infty}^2 \|D^{m-i} B\|_{L^2}^2 + C \|B\|_{L^\infty}^2 \|D^m B\|_{L^2}^2, \quad (2.20)$$

and

$$\begin{aligned} & \sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^m(B_k u_j)|^2 dx + \sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^m(u_k B_j)|^2 dx \\ & \leq C \sum_{1 \leq i \leq m/2} \|D^i B\|_{L^\infty}^2 \|D^{m-i} u\|_{L^2}^2 + C \|B\|_{L^\infty}^2 \|D^m u\|_{L^2}^2 \\ & \quad + C \sum_{1 \leq i \leq m/2} \|D^i u\|_{L^\infty}^2 \|D^{m-i} B\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \|D^m B\|_{L^2}^2. \end{aligned} \quad (2.21)$$

For the Hall term we have the estimate

$$\begin{aligned} \int_{\mathbb{R}^3} |D^m(B \cdot \nabla B)|^2 dx &= \int_{\mathbb{R}^3} |D^m \nabla \cdot (B \otimes B)|^2 dx \leq \sum_{j,k=1}^3 \int_{\mathbb{R}^3} |D^{m+1}(B_j B_k)|^2 dx \\ &\leq \|D^{m+1} B\|_{L^2}^2 \|B\|_{L^\infty}^2 + \|D^m B\|_{L^2}^2 \|\nabla B\|_{L^\infty}^2 \\ &\quad + \sum_{2 \leq i \leq \frac{m+1}{2}} \|D^i B\|_{L^\infty}^2 \|D^{m+1-i} B\|_{L^m}^2. \end{aligned} \quad (2.22)$$

Let $k \geq 3$, By the Gagliardo-Nirenberg inequality and Theorem 1.1 and Theorem 1.3 we have

$$\|B\|_{L^\infty} \leq C \|B\|_{L^2}^{\frac{2k-3}{2k}} \|D^k B\|_{L^2}^{\frac{3}{2k}} \leq C(t+1)^{-\frac{3}{4}} \quad \forall t \geq T_1, \quad (2.23)$$

for some $T_1 > 0$, where $\theta \in (0, 1)$. Hence, for any there exists $T_1 > 0$ such that $\varepsilon > 0$

$$\|D^{m+1} B(t)\|_{L^2}^2 \|B(t)\|_{L^\infty}^2 \leq \varepsilon \|D^{m+1} B(t)\|_{L^2}^2. \quad (2.24)$$

for all $t \geq T_1$. Hence, this term can be absorbed into the viscosity term. Combining (2.18)-(2.24) yields for $t \geq \max\{T_0, T_1\}$

$$\begin{aligned}
& \frac{d}{dt}(\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) + \|D^{m+1} u\|_{L^2}^2 + \|D^{m+1} B\|_{L^2}^2 \\
& \leq C_m(\|u\|_{L^\infty}^2 \|D^m u\|_{L^2}^2 + \|B\|_{L^\infty}^2 \|D^m B\|_{L^2}^2 + \|B\|_{L^\infty}^2 \|D^m u\|_{L^2}^2 \\
& \quad + \|u\|_{L^\infty}^2 \|D^m B\|_{L^2}^2 + \|\nabla B\|_{L^\infty}^2 \|D^m B\|_{L^2}^2) \\
& \quad + C_m \sum_{1 \leq j \leq m/2} (\|D^j u\|_{L^\infty}^2 \|D^{m-j} u\|_{L^2}^2 + \|D^j B\|_{L^\infty}^2 \|D^{m-j} B\|_{L^2}^2) \\
& \quad + C_m \sum_{1 \leq j \leq m/2} (\|D^j B\|_{L^\infty}^2 \|D^{m-j} u\|_{L^2}^2 + \|D^j u\|_{L^\infty}^2 \|D^{m-j} B\|_{L^2}^2) \\
& \quad + C_m \sum_{2 \leq i \leq \frac{m+1}{2}} \|D^i B\|_{L^\infty}^2 \|D^{m+1-i} B\|_{L^2}^2. \tag{2.25}
\end{aligned}$$

This completes the proof of the lemma for $m \geq 3$. Next we consider the case $m = 1, 2$. The estimate for $m = 1$ corresponding to $u \cdot \nabla u$ is the following.

$$\begin{aligned}
\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_i (u_j \partial_j u_k) \partial_i u_k dx &= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_j \partial_i \partial_j u_k \partial_i u_k dx + \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_i u_j u_k \partial_i u_k dx \\
&:= I_1 + I_2.
\end{aligned}$$

For $\varepsilon > 0$ we have

$$|I_1| \leq \int_{\mathbb{R}^3} |u| |Du| |D^2 u| dx \leq C_\varepsilon \|u\|_{L^\infty}^2 \|Du\|_{L^2}^2 + \varepsilon \|D^2 u\|_{L^2}^2.$$

Integrating by part for I_2 we have similar estimate,

$$|I_2| \leq C_\varepsilon \|u\|_{L^\infty}^2 \|Du\|_{L^2}^2 + \varepsilon \|D^2 u\|_{L^2}^2.$$

In the case $m = 2$ we have

$$\begin{aligned}
\sum_{i,j,k,m=1}^3 \int_{\mathbb{R}^3} \partial_i \partial_j (u_k \partial_k u_m) D^2 u_m dx &= \sum_{i,j,k,m=1}^3 \int_{\mathbb{R}^3} u_k \partial_i \partial_j \partial_k u_m D^2 u_m dx \\
&+ 2 \sum_{i,j,k,m=1}^3 \int_{\mathbb{R}^3} \partial_i u_k \partial_j \partial_k u_m D^2 u_m dx + \sum_{i,j,k,m=1}^3 \int_{\mathbb{R}^3} \partial_i \partial_j u_k \partial_k u_m D^2 u_m dx \\
&:= J_1 + J_2 + J_3.
\end{aligned}$$

Let $\varepsilon > 0$, we have simple estimates

$$|J_1| \leq \int_{\mathbb{R}^3} |u| |D^2 u| |D^3 u| dx \leq C_\varepsilon \|u\|_{L^\infty}^2 \|D^2 u\|_{L^2}^2 + \varepsilon \|D^3 u\|_{L^2}^2.$$

For J_2 and J_3 we obtain similar estimates after moving, by integration by part, the derivative of $\partial_i u_k$ and $\partial_k u_m$ respectively to the other factors in the integrands. Therefore we obtain

$$|J_2| + |J_3| \leq C_\varepsilon \|u\|_{L^\infty}^2 \|D^2 u\|_{L^2}^2 + \varepsilon \|D^3 u\|_{L^2}^2.$$

The estimates for the other terms corresponding to $B \cdot \nabla B, u \cdot \nabla B, B \cdot \nabla u$ are similar, and we omit them. We are left only with the Hall term. For $m = 1$ we have

$$\begin{aligned} \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i \nabla \times (B \cdot \nabla B) \cdot \partial_i B dx &= \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i (B \cdot \nabla B) \cdot \partial_i (\nabla \times B) dx \\ &= \sum_{i=1}^3 \int_{\mathbb{R}^3} (\partial_i B \cdot \nabla B) \cdot \partial_i (\nabla \times B) dx + \sum_{i=1}^3 \int_{\mathbb{R}^3} (B \cdot \nabla \partial_i B) \cdot \partial_i (\nabla \times B) dx \\ &\leq C_\varepsilon \|\nabla B\|_{L^\infty}^2 \|\nabla B\|_{L^2}^2 + \varepsilon \|D^2 B\|_{L^2}^2 + \|B\|_{L^\infty} \|D^2 B\|_{L^2}^2 \\ &\leq C_\varepsilon \|\nabla B\|_{L^\infty}^2 \|\nabla B\|_{L^2}^2 + 2\varepsilon \|D^2 B\|_{L^2}^2, \quad \forall t \geq T_1 \end{aligned}$$

where we used the fact (2.23), $\|B(t)\|_{L^\infty} \leq \varepsilon$ for all $t \geq T_1$. In the case $m = 2$ we have

$$\begin{aligned} \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \partial_i \partial_j \nabla \times (B \cdot \nabla B) \cdot D^2 B dx &= \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \partial_i \partial_j (B \cdot \nabla B) \cdot D^2 (\nabla \times B) dx \\ &= \sum_{i,j=1}^3 \int_{\mathbb{R}^3} (\partial_i \partial_j B \cdot \nabla B) \cdot D^2 (\nabla \times B) dx + 2 \sum_{i,j=1}^3 \int_{\mathbb{R}^3} (\partial_i B \cdot \nabla \partial_j B) \cdot D^2 (\nabla \times B) dx \\ &\quad + \sum_{i,j=1}^3 \int_{\mathbb{R}^3} (B \cdot \nabla \partial_i \partial_j B) \cdot D^2 (\nabla \times B) dx \\ &\leq C_\varepsilon \|\nabla B\|_{L^\infty}^2 \|D^2 B\|_{L^2}^2 + \varepsilon \|D^3 B\|_{L^2}^2 + \|B\|_{L^\infty} \|D^3 B\|_{L^2}^2 \\ &\leq C_\varepsilon \|\nabla B\|_{L^\infty}^2 \|D^2 B\|_{L^2}^2 + 2\varepsilon \|D^3 B\|_{L^2}^2 \quad \forall t \geq T_1. \end{aligned}$$

This completes the proof of the lemma for $m = 1, 2$. \square

Next lemma is more or less a repetition of Lemma 3.3 of [11].

Lemma 2.4 *Let $m \in \mathbb{N}$, $t \geq T_* = \max\{T_0, T_1\}$, where T_0, T_1 are the given in the above lemma. Assume*

$$\|D^{m-1}u\|_{L^2}^2 + \|D^{m-1}B\|_{L^2}^2 \leq C_{m-1}(t+1)^{-\rho_{m-1}} \quad \forall t \geq T_*. \quad (2.26)$$

Suppose

$$\frac{d}{dt} (\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) \leq C_0(t+1)^{-1} \|D^m u\|_{L^2}^2 + \sum_{i=1}^m C_i(t+1)^{-s_i} \quad (2.27)$$

with $s_i \geq \rho_{m-1} + 2$. Then,

$$\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2 \leq C_m(t+1)^{-\rho_m} \quad \forall t \geq T_* \quad (2.28)$$

with $\rho_m = 1 + \rho_{m-1}$, where $C_m = C_m(C_{m-1}, C_i, s_i, \rho_{m-1}, m)$ does not depend on T_ .*

Proof We use the Fourier-Splitting argument. Let

$$S = \left\{ \xi \in \mathbb{R}^3 \mid |\xi| \leq \left(\frac{C_0 + k}{t+1} \right)^{\frac{1}{2}} \right\}, \quad k = 1 + \max_{1 \leq i \leq m} \{s_i\}.$$

Then,

$$\begin{aligned} \|D^{m+1}u\|_{L^2}^2 + \|D^{m+1}B\|_{L^2}^2 &\geq \int_{S^c} |\xi|^2 (|\widehat{D^m u}|^2 + |\widehat{D^m B}|^2) d\xi \\ &\geq \frac{C_0 + k}{t+1} (\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) - \frac{C_0 + k}{t+1} \int_S (|\widehat{D^m u}|^2 + |\widehat{D^m B}|^2) d\xi \\ &\geq \frac{C_0 + k}{t+1} (\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) - \left(\frac{C_0 + k}{t+1} \right)^2 \int_S (|\widehat{D^{m-1} u}|^2 + |\widehat{D^{m-1} B}|^2) d\xi. \end{aligned}$$

Using this last inequality and the hypothesis (2.27), we have

$$\begin{aligned} \frac{d}{dt} (\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) + \frac{k}{t+1} (\|D^{m+1}u\|_{L^2}^2 + \|D^{m+1}B\|_{L^2}^2) \\ \leq C_{m-1} \frac{(C_0 + k)^2}{(t+1)^{2+\rho_{m-1}}} + \sum_{i=1}^m C_i(t+1)^{-s_i}. \end{aligned}$$

Multiplying $(t+1)^k$ and integrating in time, and dividing by $(t+1)^k$, we find

$$\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2 \leq C_m(t+1)^{-(1+\rho_{m-1})} + \sum_{i=1}^m C_i(t+1)^{-s_i+1}.$$

Since $s_i \geq \rho_{m-1} + 2$, the conclusion of the lemma follows. \square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2 We first consider the case $m = 1, 2$. From the estimate (2.23) for B and its placements by u and ∇B , and $k = 2$ we have

$$\|u(t)\|_{L^\infty}^2 + \|B(t)\|_{L^\infty}^2 + \|\nabla B(t)\|_{L^\infty}^2 \leq C(t+1)^{-\frac{3}{2}} \quad (2.29)$$

for $t \geq T_0$. Substituting (2.29) into (2.16), we obtain

$$\begin{aligned} & \frac{d}{dt}(\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) + \|D^{m+1} u\|_{L^2}^2 + \|D^{m+1} B\|_{L^2}^2 \\ & \leq C(t+1)^{-\frac{3}{2}}(\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) \end{aligned}$$

for $m = 1, 2$. We can now apply Lemma 2.4 directly to obtain (1.7). For $m \geq 3$ we need to estimate R_m of (2.17). By the Gagliardo-Nirenberg inequality we have

$$\|D^j u\|_{L^\infty} \leq C\|D^{m+1} u\|_{L^2}^{a_j} \|u\|_{L^2}^{1-a_j}, \quad (2.30)$$

and

$$\|D^j B\|_{L^\infty} \leq C\|D^{m+1} B\|_{L^2}^{a_j} \|B\|_{L^2}^{1-a_j}, \quad (2.31)$$

where $a_j = \frac{j+\frac{3}{2}}{m+1}$. Substituting (2.30) and (2.31) into R_m of (2.17), and using Young's inequality ($ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, $1/p + 1/q = 1$; $a, b > 0$; $p, q \geq 1$), we obtain

$$\begin{aligned} R_m & \leq \frac{1}{8}(\|D^{m+1} u\|_{L^2}^2 + \|D^{m+1} B\|_{L^2}^2) \\ & \quad + C(\|u\|_{L^2}^2 + \|B\|_{L^2}^2) \left(\sum_{1 \leq j \leq m/2} \|D^{m-j} u\|_{L^2}^{\frac{2}{1-a_j}} + \sum_{1 \leq j \leq m/2} \|D^{m-j} B\|_{L^2}^{\frac{2}{1-a_j}} \right) \\ & \leq \frac{1}{8}(\|D^{m+1} u\|_{L^2}^2 + \|D^{m+1} B\|_{L^2}^2) + C \sum_{1 \leq j \leq m/2} (t+1)^{-s_j}, \end{aligned} \quad (2.32)$$

where $s_j = 2\mu + \frac{(m+1)(m-j)}{m-j-1/2}$. Note that since $1 \leq j \leq m/2$ we have $s_j \geq 2\mu + m + 1$. Substituting (2.30), (2.31) and (2.32) into (2.16), we obtain the hypothesis (2.27). Applying Lemma 2.4 directly, we obtain the conclusion of the theorem for $m \geq 3$. \square

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